ture.[2] Two types of truss member cross sections are considered: a solid rod and tubular member. In addition to the existing constraints, manufacturing constraints are imposed on these members, given by

$$(r/r_{\min})-1\geq 0 \tag{23}$$

$$(t/t_{\min})-1\geq 0 \tag{24}$$

where

$r_{\min}$ = minimum outside radius of member = 0.001 m

$t_{\min}$ = minimum thickness of tubular member = 0.0004 m

Comparison of Bernoulli-Euler and Timoshenko Beam Approach

During the optimization process, the dominant constraint conditions were frequency constraint, column buckling constraint (solid members), diagonal buckling constraint, tip displacement constraint (tubular members), and minimum thickness constraint. For all of the constraint conditions except the diagonal buckling constraint, the Bernoulli-Euler beam gives almost identical results to the Timoshenko beam. However, for the critical diagonal member (adjacent to the centilevered end of the truss), the internal forces calculated using a Bernoulli-Euler beam are totally erroneous.

As an example, for a 10-bay truss, the tip deflections and diagonal member load computed using a Bernoulli-Euler and Timoshenko beam are compared to those determined using a MSC/NASTRAN truss model. The results are presented in Table 1. It is apparent that the Bernoulli-Euler beam predicts accurate tip deflections and yet totally erroneous diagonal member loads.

Comparison of Structural Efficiency

The optimized masses of the tubular member truss and the solid member truss are compared as a function of number of bays in Fig. 2. For the optimized structure, the tubular truss is quite attractive from a mass consideration as well as from the reduction in the number of bays required.

Conclusions

Through the use of equivalent modeling and simple hand calculations, an optimization of a truss structure was achieved. In addition, valuable insight was gained into the relative performance of a tubular and solid member truss structure. This approach could be extended to more complicated structures, resulting in inexpensive preliminary optimization. However, the analyst must be careful in selecting his equivalent models in order to avoid erroneous results.

References

[1]Sun, C. T., Kim, B. J., and Bogdanaff, J. L., "On the Derivation of Equivalent Simple Models for Beam- and Plate-Like Structures in Dynamic Analysis," AIAA Paper 81-0624, *Proceedings of the AIAA Dynamics Specialists Conference*, AIAA, New York, April 1981, pp. 523-532.

[2]Johnson, R. C., *Optimum Design of Mechanical Elements*, 2nd ed., Wiley, New York, 1980.

Perfect Gas Effects in Compressible Rapid Distortion Theory

E. J. Kerschen* and M. R. Myers†
University of Arizona, Tucson, Arizona

Introduction

THE interaction of unsteady disturbances with aerodynamic flows is a problem of fundamental interest, with applications to sound generation, structural vibration, and aeroelastic flutter. In the classical treatment of such problems, the mean flow is generally assumed to be uniform. However, in actual applications the mean flow gradients are often substantial. A more realistic approach is to allow the mean flow to be nonuniform, but to linearize the unsteady disturbances about this nontrivial mean flow. The analysis of linear, inviscid disturbances to irrotational mean flows has come to be known as Rapid Distortion Theory.

Hunt[1] considered the interaction of upstream vorticity disturbances with the incompressible flow around a two-dimensional body. He first determined the vorticity field and then integrated to find the induced velocities. An additional irrotational disturbance was required to satisfy the no-flow condition on the body surface. Goldstein[2] considered compressible flow and developed a simpler formulation (discussed below), which requires the solution of a single inhomogeneous wave equation.

Goldstein's inhomogeneous wave equation is linear but has variable coefficients and a complicated source term. The variable coefficients are functions of the compressible mean flow, which in general must be determined numerically. Kerschen and Balsa[3] showed that introduction of the tangent gas relations[4] allows the inhomogeneous wave equation to be transformed into a much simpler form. With the further assumption that the mean flow is a small disturbance to a uniform flow, analytical expressions could be given for the variable coefficients and source term. Unfortunately, we have found in later work that the tangent gas approximation leads to erroneous results in certain situations, as in the example included here of high-frequency gusts interacting with an airfoil. Essentially, the tangent gas approximation leads to unrealistic estimates of the variation in the speed of sound. The main purpose of this Note is to present an alternative simplified form of Goldstein's inhomogeneous wave equation that incorporates the perfect gas thermodynamic relations.

Analysis

We consider small-amplitude disturbances to a steady, irrotational, compressible mean flow. The mean flow is assumed to be uniform far upstream, and convected vortical and entropic disturbances are imposed on this uniform flow U_∞. Linearizing the equations of motion about the mean flow, and neglecting viscous and heat-conduction effects, Goldstein showed that the perturbations are described by the following equations:

$$\boldsymbol{u}'=\nabla G'+\boldsymbol{v}' \tag{1}$$

$$\frac{\mathrm{D}_0}{\mathrm{D}t}\left(\frac{1}{a_0^2}\frac{\mathrm{D}_0 G'}{\mathrm{D}t}\right)-\frac{1}{\rho_0}\nabla\cdot(\rho_0\nabla G')=\frac{1}{\rho_0}\nabla\cdot(\rho_0\boldsymbol{v}') \tag{2}$$

Received June 6, 1986.

*Associate Professor, Aerospace and Mechanical Engineering. Member AIAA.

†Research Associate, Aerospace and Mechanical Engineering.

$$p' = -\rho_0 \frac{D_0 G'}{Dt} \tag{3}$$

$$\rho' = p'/a_0^2 - \rho_0 s'/c_p \tag{4}$$

Here $\boldsymbol{u}'$, p', ρ', and s' are the fluctuating components of velocity, pressure, density, and entropy, and ρ_0 and a_0 are the density and speed of sound for the mean flow; c_p is the specific heat at constant pressure (assumed constant). The mean flow velocity is denoted by $\boldsymbol{U}$ $(=U_i)$; $D_0/Dt = \partial/\partial t + \boldsymbol{U}\cdot\nabla$ is the substantial derivative relative to the irrotational mean flow.

It is well known that the vorticity and entropy fluctuations for this flow are governed by linear, first-order partial differential equations that can be solved by the method of characteristics. Goldstein's formulation simplifies the problem by solving for the vortical velocity $\boldsymbol{v}'$ rather than for the vorticity; $\boldsymbol{v}'$ also satisfies a first-order partial differential equation and thus can be found in closed form. Explicit expressions for $\boldsymbol{v}'$ and s' in Cartesian coordinates appear in Ref. 1; the equivalent expressions in curvilinear coordinates are given below. Note that Eq. (1) does not correspond to the classical decomposition of a vector field, since in general $\boldsymbol{v}'$ is not divergence-free.

Having solved for $\boldsymbol{v}'$, Goldstein's formulation reduces the problem to the solution of a single inhomogeneous wave equation for the potential function G'. The appropriate boundary condition on any rigid surfaces that may be present is

$$\boldsymbol{n}\cdot\nabla G' = -\boldsymbol{n}\cdot\boldsymbol{v}' \tag{5}$$

where $\boldsymbol{n}$ is the normal to the surface. The radiation or outgoing-wave condition applies at infinity.

We consider a two-dimensional mean flow, i.e., $U_i = U_i(x_1,x_2)$, $i=1,2$ and $U_3=0$. Since the vorticity and entropy fluctuations are convected along the mean flow streamlines, it is convenient to introduce the velocity potential ϕ and the stream function ψ of this flow as orthogonal coordinates in the (x_1,x_2) plane. The appropriate coordinate metrics are $h_\phi = 1/U_0$ and $h_\psi = \rho_\infty/\beta_\infty\rho_0 U_0$, where U_0 is the magnitude of $\boldsymbol{U}$. The substantial derivative becomes $D_0/Dt = (\partial/\partial t + U_0^2\partial/\partial\phi)$. For convenience, we also introduce $z = U_\infty x_3$.

Since the governing equations are linear and the mean flow is uniform far upstream, an arbitrary upstream disturbance can be represented as a superposition of harmonic waves. Hence we consider vortical and entropic disturbances having the following form far upstream:

$$\begin{bmatrix} \boldsymbol{v}' \\ s' \end{bmatrix} = \begin{bmatrix} U_\infty \boldsymbol{A} \\ 2c_p B \end{bmatrix} \exp[i(k_t\phi + k_n\psi + k_z z - k_t U_\infty^2 t)] \tag{6}$$

The expressions for the vortical velocity and entropy are then given by[3]

$$\boldsymbol{v}'(\phi,\psi,z,t) = (v_t, v_n, v_z)\exp[i(k_z z - k_t U_\infty^2 t)] \tag{7a}$$

$$v_t/U_\infty = (A_t^* U_\infty/U_0 + BU_0/U_\infty)e^{i\sigma} \tag{7b}$$

$$v_n/U_\infty = (\rho_0 U_0/\rho_\infty U_\infty)[A_n + \beta_\infty A_t^* \partial g/\partial\psi]e^{i\sigma} \tag{7c}$$

$$v_z/U_\infty = A_z e^{i\sigma} \tag{7d}$$

$$s' = 2c_p B e^{i\sigma} \tag{7e}$$

where

$$\sigma = k_t\phi + k_t g(\phi,\psi) + k_n\psi, \qquad A_t^* = A_t - B$$

$$g(\phi,\psi) = \int_{-\infty}^{\phi}\left[\frac{U_\infty^2}{U_0^2(\zeta,\psi)} - 1\right]d\zeta \tag{7f}$$

The function $g(\phi,\psi)$ is essentially Lighthill's Drift function, which is related to the distortion of fluid material lines.

We next consider Eq. (2) for G. Introducing the curvilinear coordinates and factoring out the harmonic dependence on z and t,

$$G' = G\exp[i(k_z z - k_t U_\infty^2 t)] \tag{8a}$$

we obtain[3]

$$\begin{aligned} &\frac{\partial}{\partial\phi}\left(\beta_0^2\frac{\partial G}{\partial\phi}\right) + \frac{\partial}{\partial\psi}\left(\frac{\beta_\infty^2\rho_0^2}{\rho_\infty^2}\frac{\partial G}{\partial\psi}\right) + 2ik_t\left(\frac{U_\infty^2}{a_0^2}\frac{\partial G}{\partial\phi}\right) \\ &+ \frac{U_\infty^2}{a_0^2}\left[\frac{k_t^2 U_\infty^2}{U_0^2} - \frac{k_z^2}{M_0^2} - 2ik_t\frac{\partial}{\partial\phi}(\log a_0)\right]G \\ &= -\frac{\partial}{\partial\phi}\left(\frac{U_\infty}{U_0}v_t\right) - \frac{\partial}{\partial\psi}\left(\frac{\rho_0 U_\infty\beta_\infty}{\rho_\infty U_0}v_n\right) - ik_z\frac{U_\infty^2}{U_0^2}v_z \end{aligned} \tag{8b}$$

where M_0 is the local Mach number of the mean flow, and $\beta_0^2 = (1-M_0^2)$.

For a compressible flow, the mean flow quantities that appear in Eq. (8b) must be determined numerically, which precludes any possibility of an analytical solution for G. Thus it is worthwhile to consider simplified forms of this equation. Kerschen and Balsa[3] explored the simplifications that arise from the tangent gas approximation, which consists of a linearization of the perfect gas equation of state in the pressure-specific volume plane. For a tangent gas $a_0/a_\infty = \rho_\infty/\rho_0$, and this simplified equation of state allows Eq. (8b) to be transformed into a much simpler form. However, the evaluation of the Drift function $g(\phi,\psi)$ still requires numerical methods. A closed-form approximation for the Drift function can be obtained with the further assumption that the mean flow is a small perturbation to a uniform flow. We require that the mean flow perturbation [say $\mathcal{O}(\alpha)$] be much larger than the $\mathcal{O}(\epsilon)$ unsteady disturbances.

Unfortunately, the tangent gas theory does not accurately represent the variations in speed of sound associated with most fluids of practical interest. The speed of sound for a perfect gas is given by

$$a_0/a_\infty = [\rho_0/\rho_\infty]^{(\gamma-1)/2}, \qquad \gamma = c_p/c_v$$

Hence, a tangent gas is a perfect gas with $\gamma = -1$. For comparison, the value of γ for air is approximately 1.4. This difference in γ can significantly affect the predicted features of the unsteady flow. For example, consider acoustic propagation through a mean flow with velocity and pressure variations. For a perfect gas with $\gamma = 1.4$, the speed of sound decreases as the pressure (and density) decrease, whereas for a tangent gas the speed of sound increases under these circumstances.

Thus, here we develop a simplified form of Eq. (8b) which corresponds to a perfect gas mean flow. In order to obtain closed-form expressions for the coefficients and source term, we assume that the mean flow is a small perturbation [say $\mathcal{O}(\alpha)$, $\alpha \gg \epsilon$] to a uniform flow.

The small-perturbation, perfect-gas relations for the mean flow are

$$\frac{U_0}{U_\infty} = 1 + q \qquad \text{where} \quad q = \mathcal{O}(\alpha) \tag{9a}$$

$$\frac{a_0}{a_\infty} = 1 - \frac{\gamma-1}{2}M_\infty^2 q \tag{9b}$$

$$\frac{M_0}{M_\infty} = 1 + \left(1 + \frac{\gamma-1}{2}M_\infty^2\right)q \tag{9c}$$

$$\frac{\beta_0}{\beta_\infty} = 1 - \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right) \frac{M_\infty^2}{\beta_\infty^2} q \tag{9d}$$

$$\frac{\rho_0}{\rho_\infty} = 1 - M_\infty^2 q \tag{9e}$$

Using these relations in Eq. (8b), making the following generalized Miles transformation,

$$\begin{aligned} h &= G\exp(ik_t M_\infty^2 \phi/\beta_\infty^2)\exp(-M_\infty^2 q) \\ &= G\exp(ik_t M_\infty^2 \phi/\beta_\infty^2)\,[1 - M_\infty^2 q + \mathcal{O}(\alpha^2)] \end{aligned} \tag{10a}$$

and neglecting terms of $\mathcal{O}(\alpha^2)$, we obtain after a considerable amount of manipulation

$$\begin{aligned} &\frac{\partial^2 h}{\partial \phi^2} + \frac{\partial^2 h}{\partial \psi^2} + w^2(1 - 2\beta_\infty^2 q)h \\ &+ (\gamma+1)\frac{M_\infty^4}{\beta_\infty^2}\left\{q\left[\frac{\partial^2 h}{\partial \psi^2} + 2i\delta\frac{\partial h}{\partial \phi} + (w^2+\delta^2)h\right]\right. \\ &\left. - \frac{\partial q}{\partial \phi}\left[\frac{\partial h}{\partial \phi} - i\delta h\right]\right\} = S(\phi,\psi)e^{i\Omega} \end{aligned} \tag{10b}$$

where

$$\delta = k_t/\beta_\infty^2, \qquad w^2 = (M_\infty \delta)^2 - (k_z/\beta_\infty)^2$$

$$\Omega = \delta\phi + k_n\psi + k_t g(\phi,\psi)$$

and

$$\begin{aligned} S(\phi,\psi) = 2\Bigg[& i\left(\frac{k_t A_t^*}{\beta_\infty^2} - k_n A_n \beta_\infty\right)q + i\left(k_n A_t^* + \frac{k_t A_n}{\beta_\infty}\right)\mu \\ & + \frac{A_t^* M_\infty^2}{\beta^2_\infty}\frac{\partial q}{\partial \phi} + \frac{A_n M_\infty^2}{\beta_\infty}\frac{\partial q}{\partial \psi}\Bigg] \end{aligned} \tag{10c}$$

Here $\beta_\infty \mu(\phi,\psi)$ is the mean flow angle relative to the uniform flow at upstream infinity. The Drift function $g(\phi,\psi)$ is given by

$$g(\phi,\psi) = -2Re\left\{F\left(\frac{\phi + i\psi}{U_\infty}\right)\right\} \tag{10d}$$

where $F(z')$ is the complex potential found by applying a Prandtl-Glauert transformation

$$z' = x_1' + ix_2' = x_1 + i\beta_\infty x_2$$

to the equations describing the perturbation flow in the physical (x_1, x_2) plane. The arbitrary constant in $F(z')$ is chosen such that the Drift function vanishes at upstream infinity. The functions $q(\phi,\psi)$ and $\mu(\phi,\psi)$ are also given by $\mathrm{d}F/\mathrm{d}z' = U_\infty(q - i\mu)$. The transformed boundary condition is

$$\begin{aligned} &\left[\frac{\partial h}{\partial \psi} + M_\infty^2 \frac{\partial q}{\partial \psi} h\right]_{\psi=\psi_0} \\ &= -\left[\frac{A_n}{\beta_\infty}(1 - M_\infty^2 q) - 2A_t^* \mu\right]e^{i\Omega} \end{aligned} \tag{10e}$$

where $\psi = \psi_0$ on the body surface, and the range of ϕ corresponds to the body length in the (ϕ,ψ) plane. Equations (10) represent the final form of our governing equations for the modified unsteady disturbance potential $h(\phi,\psi)$.

Discussion

The tangent gas theory was originally developed as an approximate model for compressibility effects in steady, subsonic aerodynamic calculations. A good discussion can be found in Ref. 5, where it is argued that the tangent gas model is significantly more accurate than Rayleigh-Janzen or Prandtl-Glauert compressibility corrections. However, this advantage does not carry over to unsteady flows, at least in the high-frequency limit.

The tangent gas equations governing the unsteady motion for a small-perturbation mean flow may be obtained by setting $\gamma = -1$ in Eq. (10b). In comparison, it is evident that the different operator in the perfect gas theory is considerably more complicated. The special relations between density, speed of sound, and Mach number for a tangent gas result in the elimination of all of the variable coefficients of the derivatives on the modified potential h. Since the tangent gas assumptions do not affect the mean flow velocity or density, the expressions for the vortical velocity, the source term, and the boundary condition are identical in perfect gas and tangent gas formulations.

Regarding the applicability of the tangent gas approximation, the penalty paid for wrong prediction of the sound speed (or rather the first-order correction to it) can outweigh the advantage of simplicity. This is the case if the irrotational field arising as a solution of Eqs. (10) is a high-frequency acoustic wave. For example, in the authors' investigation of the interaction of a convected disturbance with an airfoil at an angle of attack,[6] acoustic waves are generated at the leading edge of the airfoil. These waves propagate along the airfoil (assumed long compared to the acoustic wavelength) and are scattered by the trailing edge. The strength of the scattered field is determined by the relative phases of the waves above and below the airfoil. As the following calculation demonstrates, tangent gas theory significantly overestimates the phase shift at the trailing edge, which results in a scattered field strength that is too small.

We take the airfoil to be a flat plate at a small angle of attack α to the mean flow, and occupying the region $0 < x_1 < \ell$, $x_2 = 0$. (Cartesian coordinates are simplest here.) The phase of an acoustic signal at the trailing edge of the airfoil (assuming the phase is zero at the leading edge) is given by

$$\text{phase}(TE) = \int_0^\ell \frac{\omega}{U_0(x) + a_0(x)}\,\mathrm{d}x \tag{11a}$$

where ω is the frequency of the wave. Expanding the mean flow velocity and sound speed into their small-perturbation $[\mathcal{O}(\alpha)]$ perfect gas expansions and ignoring second-order terms, we obtain

$$\begin{aligned} \text{phase}(TE) = &\left(\frac{\omega\ell}{U_\infty}\right)\frac{M_\infty}{1+M_\infty}\left\{1 - \frac{M_\infty}{1+M_\infty}\right. \\ &\left.\times\left[1 - \frac{\gamma-1}{2}M_\infty\right]\frac{1}{\ell}\int_0^\ell q\,\mathrm{d}x\right\} \end{aligned} \tag{11b}$$

Utilizing thin airfoil theory to evaluate q and integrating, we obtain

$$\begin{aligned} \text{phase}(TE) = &\left(\frac{\omega\ell}{U_\infty}\right)\frac{M_\infty}{1+M_\infty}\left\{1 \mp \frac{M_\infty}{1+M_\infty}\right. \\ &\left.\times\left[1 - \frac{\gamma-1}{2}M_\infty\right]\frac{\pi}{2}\frac{\alpha}{\beta_\infty}\right\} \end{aligned} \tag{11c}$$

where the minus and plus signs apply on the upper and lower surfaces, respectively.

The amplitude of the scattered wave from the trailing edge depends on the pressure jump, across the trailing edge, produced by the leading-edge acoustic field discussed previously. Subtracting complex exponentials with the preceding phases, we find that the scattered wave amplitude is proportional to

$$\cos\left\{\left(1-\frac{\gamma-1}{2}M_\infty\right)\left(\frac{M_\infty}{1+M_\infty}\right)^2\frac{\omega\ell}{U_\infty}\frac{\pi\alpha}{2\beta_\infty}\right\}$$

When $M_\infty=0.80$, $\alpha=6$ deg, and the high-frequency parameter $\omega\ell/U_\infty=10$, this factor is 0.90 for a perfect gas ($\gamma=1.4$) and 0.56 for a tangent gas. Thus the amplitude of the scattered wave is considerably underestimated by tangent gas theory. This effect is even more pronounced for larger Mach numbers and higher frequencies. In addition to the error in scattered wave amplitude, the tangent gas theory also produces gross phase errors in the far field.

In summary, we have derived the governing equations for small-amplitude unsteady disturbances imposed on steady, compressible mean flows that are two-dimensional and nearly uniform. The present equations are based on the perfect gas equation of state and hence generalize previous results based on tangent gas theory. The new equations are more complex, but this additional complexity is necessary to properly describe high-frequency disturbances, particularly when the base flow Mach number is large. Under these circumstances, the simplifying assumptions of tangent gas theory are not applicable.

Acknowledgment

The work described herein was funded mainly by NASA Lewis Research Center under Grant NAG3-357. Partial support was also contributed by General Electric Corporate Research and Development. We thank Professor T. F. Balsa for many stimulating and productive technical discussions.

References

[1]Hunt, J. C. R., "A Theory of Turbulent Flow Round Two-Dimensional Bluff Bodies," *Journal of Fluid Mechanics*, Vol. 61, Dec. 1973, pp. 625–706.

[2]Goldstein, M. E., "Unsteady Vortical and Entropic Distortions of Potential Flows Round Arbitrary Obstacles," *Journal of Fluid Mechanics*, Vol. 89, Dec. 1978, pp. 433–468.

[3]Kerschen, E. J. and Balsa, T. F., "Transformation of the Equation Governing Disturbances of a Two-Dimensional Compressible Flow," *AIAA Journal*, Vol. 19, Oct. 1981, pp. 1367–1370.

[4]Woods, L. C., *The Theory of Subsonic Plane Flow*, Cambridge University Press, London, 1961.

[5]Lighthill, M. J., "Higher Approximations," *General Theory of High Speed Aerodynamics*, Vol. VI of *High Speed Aerodynamics and Jet Propulsion*, edited by W. R. Sears, Princeton University Press, Princeton, NJ, 1954, pp. 345–489.

[6]Kerschen, E. J. and Myers, M. R., "Influence of Incidence Angle on Sound Generated by Airfoils Interacting with High-Frequency Convected Gusts," *Journal of Fluid Mechanics* (submitted for publication).